Recent Progress on the Geometry of Univalence Crtiteria

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1 Introduction

This paper is a survey of some new techniques and new results on sufficient conditions in terms of the Schwarzian derivative for analytic functions defined in the unit disk to be univalent. Along with univalence we consider the questions of quasiconformal and homeomorphic extensions of the mapping.

Let f be analytic and locally univalent. Its Schwarzian derivative is

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

If $u = (f')^{-1/2}$ then

$$u'' + \frac{1}{2}(Sf)u = 0.$$

Conversely if u is a solution of

$$u'' + pu = 0 \tag{1.1}$$

and

$$f(z) = \int_{z_0}^{z} u^{-2}(\zeta)\, d\zeta \tag{1.2}$$

then $Sf = 2p$. We recall the chain rule

$$S(f \circ g) = ((Sf) \circ g))(g')^2 + Sg \tag{1.3}$$

and that the Schwarzian is identically zero exactly for Möbius transformations. Let $\mathbf{D}$ denote the unit disk.

There has been progress in several areas, but the innovations we treat here come primarily from an injectivity criterion for conformal, local diffeomorphisms of an n-dimensional Riemannian manifold into the n-sphere. The criterion involves a generalization of the Schwarzian derivative which depends both on the conformal factor of the mapping and on the underlying Riemannian metric. The scalar curvature of the metric and the metric diameter of the manifold enter as bounds for the Schwarzian. A majority of the known classical univalence criteria follow from this general result.

The proof of the general criterion synthesizes several key ingredients that are present in the proofs of many classical criteria, most particularly the Sturm comparison theorem for second order

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ordinary differential equations, applied to (1.1). Comparison theorems can be considered as statements about convexity, and this turns out to be the essential notion through which we would like to understand and explain the phenomena of univalence and extensions.

2 The Nehari Class and the Ahlfors-Weill Extension

Nehari's theorem of 1949 in [11] is perhaps the best known univalence criterion, and its connection to quasiconformal mappings discovered in 1962 by Ahlfors and Weill [1] are together the model cases for other classical crtiteria and their generalizations. Instead of beginning with the very general formulations now available we would therefore like to highlight the new ideas in the context of these two fundamental results.

Theorem 1 (Nehari, Ahlfors-Weill) *If f is analytic in $\mathbf{D}$ and*

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}, \quad z \in \mathbf{D}, \tag{2.1}$$

then f is univalent. If

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2}, \quad z \in \mathbf{D}, \tag{2.2}$$

for some $0 \leq t < 1$ then f has a $\frac{1+t}{1-t}$-quasiconformal extension to $\widehat{\mathbf{C}} = \mathbf{C} \cup \infty$.

We let N be the set of all (univalent) functions that satisfy (2.1) and we call it the Nehari Class. N is a large class of functions, containing, for example, all convex conformal mappings, *i.e.*, those mappings with a (euclidean) convex image, [17], [12].

There is an interpretation of (2.1) in terms of convexity in hyperbolic geometry. Let $f \in N$, $\Omega = f(\mathbf{D})$ and let $\lambda_\Omega|dw|$ be the Poincaré metric of Ω. From calculations to be discussed in the next section – based essentially on a generalization of the differential equation (1.1) – it follows that $\lambda_\Omega^{1/2}$ is a hyperbolically convex function on Ω. This means that $\lambda_\Omega^{1/2}$ is convex, in the usual sense, along all hyperbolic geodesics in Ω, or equivalently that the Hessian of $\lambda_\Omega^{1/2}$, computed with respect to the Poincaré metric, is positive definite. Since (2.1) is invariant under Möbius transformations of the image, this convexity property is true for all Möbius shifts, $M(\Omega)$. This Möbius invariant hyperbolic convexity turns out to be equivalent to (2.1).

Nehari's proof related the univalence of f to the disconjugacy of solutions of (1.1). An upper bound on the Schwarzian allows one to invoke variants of the Sturm comparison theorem. Normalizing via Möbius transformations of the range and domain are possible because of the chain rule (1.3) and Schwarz's lemma, according to which

$$(1-|w|^2)^2|Sf(w)| = (1-|z|^2)^2|S(f \circ g)(z)|, \quad w = g(z) = e^{i\theta}\frac{z-\zeta}{1-\bar{\zeta}z}.$$

There were three elements in the proof of the Ahlfors-Weill result. First, one can consider $f(rz)$, $r < 1$, and assume that a function satisfying (2.2) is analytic on $\overline{\mathbf{D}}$. This is then later removed by taking a limit as $r \to 1$. Second, and most remarkable, is an explicit formula for the extension:

$$E_f(z) = f(z) \quad \text{for } |z| \leq 1, f(\zeta) + \frac{(1-|\zeta|^2)f'(\zeta)}{\bar{\zeta} - \frac{1}{2}(1-|\zeta|^2)\frac{f''}{f'}(\zeta)}, \quad \zeta = 1/\bar{z}, \quad \text{for } |z| > 1. \tag{2.3}$$

The complex dilatation of E_f at a point z outside the disk is

$$\mu_{E_f}(z) = -\frac{1}{2}\left(\frac{\zeta}{\bar{\zeta}}\right)^2 (1-|\zeta|^2)^2 Sf(\zeta), \quad \zeta = 1/\bar{z}. \tag{2.4}$$

From this and (2.2) the complex dilatation is bounded by $t < 1$, and E_f will be proved to be quasiconformal once we know that it is a homeomorphism. This is then the third point in the argument, for (2.2) also imples that E_f is a local homeomorphism. Its range is $\widehat{\mathbf{C}}$ and hence by the Monodromy Theorem it is a global homeomorphism. Note that regularity on $\overline{\mathbf{D}}$ is essential in analyzing the continuity of the extension at the boundary. Note also that the proof gives no information on what happens to the extension in the limiting case as $t \to 1$.

The steps in the Ahlfors-Weill proof have been superceded, and the extension E_f does more than advertized. To explain, we first recall some results from an important paper of Gehring and Pommerenke in 1985, [10].

We adopt the terminology from [5] and say that f is an *extremal function* for (2.1) if $f(\mathbf{D})$ is not a Jordan domain. This is the case for the mapping

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

onto a parallel strip, for which $SL(z) = 2/(1-z^2)^2$. The image is not a Jordan domain, and neither is $(A \circ L \circ B)(\mathbf{D})$ for any Möbius transformation A and any Möbius automorphism B of $\mathbf{D}$. Such Möbius conjugates of L are the only extremals in N:

Theorem 2 (Gehring-Pommerenke) *If $f \in N$ then f has a continuous extension to $\overline{\mathbf{D}}$. The function L is the unique extremal function in N up to Möbius transformations of the range and of $\mathbf{D}$ onto itself.*

By suitable normalizations and use of comparison theorems one can establish a logarithmic modulus of continuity for f, and this suffices to prove the continuity up to the boundary. This replaces the first part of the Ahlfors-Weill argument, and we will add some further remarks in Section 3.

We let N^* be the set N minus all Möbius conjugations of the extremal L. Thus if $f \in N^*$ then $f(\mathbf{D})$ is a Jordan domain, and so, for topological reasons, has a homeomorphic extension to $\widehat{\mathbf{C}}$. In fact, (2.3) already gives a homeomorphic extension when $f \in N^*$, that is for $t = 1$ in (2.2). This is the main new result in this section and we explain now where it comes from.

Let $f \in N$ and let $\Omega = f(\mathbf{D})$. Again we let $\lambda_\Omega|dw|$ be the Poincaré metric of Ω. Define

$$\Lambda_\Omega(w) = w + \frac{1}{\partial_w \log \lambda_\Omega(w)}, \tag{2.5}$$

where

$$\partial_w = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right).$$

Thus $\Lambda_\Omega(w)$ moves away from w by a vector in the direction $\nabla \log \lambda_\Omega(w)$ of magnitude $2/|\nabla \log \lambda_\Omega(w)|$. When $f \in N^*$ we will show that Λ_Ω is a reflection across $\partial\Omega$; we think of it as a *gradient reflection.* In terms of Λ_Ω the Ahlfors-Weill extension is

$$E_f(z) = f(z) \quad \text{for } |z| \le 1, \Lambda_\Omega(f(\zeta)), \quad \zeta = 1/\bar{z}, \quad \text{for } |z| > 1. \tag{2.6}$$

Whatever mapping properties (2.6) is to enjoy must already be present in Λ_Ω. For example, if Λ_Ω is to be injective, then the Poincaré density λ_Ω must have at most one critical point, because such a point will be mapped under Λ_Ω to the point at infinity.

Λ_Ω has the important property of being *conformally natural.* This means that

$$\Lambda_{M(\Omega)}(M(w)) = M(\Lambda_\Omega(w)), \quad w \in \Omega \tag{2.7}$$

for any Möbius transformation M. We conclude from this that for Λ_Ω to be injective the density of the Poincaré metric must have at most one critical point for every shift of the image by a Möbius transformation. This goes the other way too, for if $\Lambda_\Omega(w_1) = \Lambda_\Omega(w_2)$ for two distinct points then some Möbius transformation takes this common value to infinity, showing that the density of the Poincaré metric of the shifted domain has at least two critical points.

Next, suppose that λ_Ω has a unique critical point. We claim that Ω is bounded. The proof of this uses the fact that the hyperbolic convexity of $\lambda_\Omega^{1/2}$ in Ω is equivalent to the function

$$u_f(z) = \frac{1}{\sqrt{|f'(z)|(1-|z|^2)}} \tag{2.8}$$

being hyperbolically convex in $\mathbf{D}$. The corresponding critical point in $\mathbf{D}$ may be assumed to be the origin. We thus have a convex function with a unique critical point, whose growth is therefore at least linear. This gives a lower bound in (2.8), and the resulting upper bound for $|f'|$ is of the form

$$|f'(z)| \le \frac{(1-|z|^2)}{(a+bL(|z|))^2} = -\frac{d}{d|z|}\left(\frac{1}{b(a+bL(|z|))}\right),$$

where a and b are positive constants depending on f. This shows that f is bounded. So Ω is bounded and the unique critical point is sent by Λ_Ω to infinity. This is an interesting global conclusion to draw. The Möbius invariant formulation of this result is that, if for every Möbius shift of Ω the density of the Poincaré metric has at most one critical point, then Λ_Ω takes values in the complement of $\overline{\Omega}$.

What about the boundary behavior of Λ_Ω? Again, the assumption is the absence of multiple critical points in every Möbius shift. A convex function with a unique critical point not only has at least linear growth, but its radial derivative moving away from the critical point must be bounded away from zero outside some small disk centered at the point. Since all is computed in the hyperbolic metric, this means that

$$\frac{1}{\lambda}|\partial_w \lambda_\Omega^{1/2}| \ge c > 0$$

away from the critical point. Thus

$$|\partial_w \log \lambda_\Omega| \ge 2c\lambda_\Omega^{1/2}, \tag{2.9}$$

which proves that Λ_Ω is the identity at the boundary because $\lambda_\Omega \to \infty$ there when the domain is bounded. We mention in passing that the exponent $1/2$ in the estimate above is sharp for functions in N^*.

To summarize, starting with $f \in N$, $\Omega = f(\mathbf{D})$, if the density of Poincaré metric of every Möbius shift of Ω has at most one critical point, then Λ_Ω is a reflection across $\partial\Omega$. When does this happen? Suppose that for some Möbius shift of Ω there are two critical points. By convexity, the hyperbolic geodesic joining them must consist entirely of critical points. Back in $\mathbf{D}$ that geodesic can be taken to lie along the real axis, and it is then easy to see that the original function f must be Möbius conjugate to the logarithm L. We can therefore now state, [4]:

Theorem 3 *If $f \in N^*$ then the Ahlfors-Weill extension E_f is a homeomorphism of the sphere to itself.*

The Ahlfors-Weill theorem itself follows immediately by virtue of the formula (2.4) for the complex dilatation.

Notes Using comparison theorems requires not just the differential equation $u'' + qu = 0$ but also initial conditions. Typically one sets up the problem, in the disk, with $u(0) = 1$ and $u'(0) = 0$. With f as in (1.2) one then has $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. In terms of the Poincaré metric the condition $f''(0) = 0$ translates exactly to $\log \lambda_\Omega$ having a critical point at $f(0)$.

See [3] for some uses of comparison theorems to prove distortion theorems for the class N. One useful fact that comes out of that work is that it is possible to normalize a function $f \in N$, by a Möbius transformation, to have $f''(0) = 0$ without introducing a pole. We also refer to [7] and [6] for further mapping properties of functions in the Nehrai class and its generalizations, especially in relation to John domains. We will not review the particular results here because we want to stay close to the theme of using convexity, and concentrate on the general geometric methods. See also [13] for more background on the Schwarzian.

3 A General Univalence Criterion: Extensions and Extremals

The title of this section comes from the paper [5]. Our intention is now to show how univalence, homeomorphic and quasiconformal extensions and convexity play out in a more general setting. We work with conformal metrics on $\mathbf{D}$ and with a generalized Schwarzian derivative.

Let g be a metric (tensor) on $\mathbf{D}$ and let $g_0 = |dz|^2$ denote the euclidean metric. For a smooth, real-valued function ψ on $\mathbf{D}$ we define a symmetric, traceless 2-tensor

$$B_g(\psi) = \operatorname{Hess}_g \psi - d\psi \otimes d\psi - \frac{1}{2}\left\{\Delta_g \psi - \|\operatorname{grad}_g \psi\|_g{}^2\right\} g.$$

The Hessian, gradient, Laplacian, and norm are computed with respect to g. If $f\colon (\mathbf{D}, g) \to (\mathbf{C}, g_0)$ is a conformal, local diffeomorphism with $f^* g_0 = e^{2\psi} g$, its *Schwarzian tensor* is defined by

$$\mathcal{S}_g f = B_g(\psi).$$

This definition is in [15], see [16] for another approach. For the present discussion it is less important to know all the aspects of the generalization than simply to keep in mind that the Schwarzian tensor is computed with respect to a background metric g, and that it changes when g changes. When there is conformal change in g the Schwarzian tensor changes in a simple way, governed by a generalization of the chain rule, (1.3). When g is the euclidean metric $\mathcal{S}_g f$ can be written as the matrix

$$\mathcal{S}_g f = (\operatorname{Re} Sf - \operatorname{Im} Sf - \operatorname{Im} Sf - \operatorname{Re} Sf).$$

In [14] the authors obtained a general univalence criterion in terms of $\mathcal{S}_g f$ that involves both the curvature of the metric and a diameter term. Let $K(g)$ denote the Gaussian curvature of the metric g. In the two-dimensional case the result can be stated as:

Theorem 4 *Let f be analytic or meromorphic in $(\mathbf{D}, g)$ and locally univalent. Suppose that any two points in $\mathbf{D}$ can be joined by a geodesic of length $< \delta$, for some $0 < \delta \le \infty$. If*

$$||\mathcal{S}_g f||_g \le \frac{2\pi^2}{\delta^2} - \frac{1}{2} K(g) \tag{3.1}$$

then f is univalent.

Many known criteria for univalence follow from Theorem 4 by choosing different conformal background metrics g. Nehari's criterion (2.1) results by taking g to be $|dz|^2/(1-|z|^2)^2$, the Poincaré metric, with $K=-4$ and $\delta=\infty$. As it turns out, the Schwarzian of f is the same in the hyperbolic and euclidean metrics, and in computing the hyperbolic norm $||\mathcal{S}_g f||_g$ one gets two factors of $(1-|z|^2)$ because $\mathcal{S}_g f$ is a 2-tensor.

There is a second order differential equation that goes along with $\mathcal{S}_g f$ involving the Hessian of a function associated to the conformal factor of f. The use of comparison theorems is then still very much central to the analysis, though the geometric set-up has changed.

We consider metrics of the form

$$g = e^{2\sigma} g_0.$$

Let f be a conformal, local diffeomorphism of $(\mathbf{D}, g)$ into $(\mathbf{C}, g_0)$. If we write $\varphi = \log|f'|$ then $f^* g_0 = e^{2(\varphi-\sigma)} g$, and hence $\mathcal{S}_g f = B_g(\varphi - \sigma)$. We define

$$u_f = e^{(\sigma-\varphi)/2}. \tag{3.2}$$

We refer to u_f as the *associated function.* Note that if we use the euclidean metric in both the domain and the range of f then $u_f = |f'|^{-1/2}$.

A calculation, straightforward but somewhat involved, gives the following:

Theorem 5 *If* $||\mathcal{S}_g f||_g \leq \dfrac{2\pi^2}{\delta^2} - \dfrac{1}{2}K(g)$ *then* $\mathrm{Hess}_g\, u_f + \dfrac{\pi^2}{\delta^2} u_f\, g \geq 0.$

Analyzing the Hessian equation along geodesics is the main part of the proof of Theorem 4.

If the metric g is complete then $\delta = \infty$ and we only consider metrics of nonpositive curvature. We write the condition (3.1) in the form

$$||\mathcal{S} f_g||_g \leq \frac{1}{2}|K(g)|. \tag{3.3}$$

The conclusion of Theorem 5 is then that

$$\mathrm{Hess}_g\, u_f \geq 0,$$

that is, u_f is a convex function on $\mathbf{D}$ with respect to the metric g. It may not look like it, but in the case of the Nehari class this is precisely the statement that the function u_f in (2.8) is hyperbolically convex in $\mathbf{D}$, or that $\lambda_\Omega^{1/2}$ is hyperbolically convex on $\Omega = f(\mathbf{D})$, $f \in N$.

As with the Nehari class, the convexity is a characteristic property for functions to satisfy (3.3). And again the nature of the critical points of u_f have consequences that are global:

Theorem 6 *Let g be complete and f a meromorphic function in $(\mathbf{D}, g)$. Then $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$ if and only if $u_{M\circ f}$ is convex for all Möbius transformations M. If u_f has a critical point at which u_f is positive, then f is analytic. If the critical point is unique then f is bounded.*

Extension to the Boundary We assume that g is complete. A typical way to use the convexity of u_f is the following. Let γ be a geodesic in the metric g in the disk, parametrized by arclength $\tau = d_g(z, z_0)$, where $z_0 \in \gamma$ is a fixed point and $d_g(z, z_0)$ is distance. Since the metric is complete, starting at z_0 we can follow γ in both directions indefinitely. Let γ_+, γ_- denote these two halves. Suppose that the derivative of u_f at z_0 in the direction of γ_+ is positive, say $b > 0$, Convexity implies that u_f will grow at least linearly along γ_+, that is,

$$u_f(z) \geq a + bd_g(z, z_0),$$

or

$$|f'(z)| \leq \frac{e^\sigma}{(a + bd_g(z, z_0))^2} = -\frac{d}{ds}\frac{1}{b(a + bd_g(z, z_0))}, \tag{3.4}$$

where s is the euclidean arclength. This implies that the integral

$$\int_{\gamma_+} |f'(z)||dz| \tag{3.5}$$

is finite, which implies that $f(z)$ has a limit as z tends to the boundary ∂D along γ_+.

The finiteness of the integral (3.5), together with a very natural geometric condition on the behavior of the geodesics near the boundary, allow us to generalize to this setting the result of Gehring and Pommerenke regarding continuous extension to the closed disk. The necessary geometric condition, which we must take as an assumption on the metric, appears frequently in areas of differential geometry concerned with the 'visibility' of a boundary at infinity. We formulate it as follows:

Definition The complete metric g on $\mathbf{D}$ has the *Unique Limit Point* property (ULP) if:
(a) Let $z_0 \in \mathbf{D}$. If $\gamma(t)$, $0 \leq t < T \leq \infty$ is a maximally extended geodesic starting at z_0 then $\lim_{t \to T} \gamma(t)$ exists (in the euclidean sense). We denote it by $\gamma(T) \in \partial\mathbf{D}$.
(b) The limit point is a continuous function of the initial direction at z_0.
(c) Let $\zeta \in \partial\mathbf{D}$. Then there is a geodesic starting at z_0 whose limit point on $\partial\mathbf{D}$ is ζ.

The problem of extending the map to $\overline{\mathbf{D}}$ is local, and we work on sectors defined by a vertex z_0 near the boundary and all geodesics emanating from z_0 toward the boundary. For small enough angle of the sector at z_0, and with a proper normalization of the mapping f, we can achieve that the derivative of u_f at z_0 in the direction of any of the geodesics is (uniformly) positive. This ensures at least a linear growth of u_f along each geodesic. By (3.4) and (3.5) all integrals

$$\int_{\gamma_+} |f'(z)||dz|$$

for γ_+ in the sector will be finite. The limit $\lim_{z \in \gamma_+} f(z)$ must lie on the boundary of the image, and it is not difficult to show that it depends continuously on the direction of the geodesic γ_+ at z_0. This shows that (locally) the boundary of the image can be continuously parametrized, which ensures the continuous extension of the mapping f.

This result also holds when the metric is not complete, but the argument is more involved. We mention also that in specific cases, the bound for $|f'|$ obtained from (3.4) can be used to show directly that the mapping f is Hölder continuous, or even Lipschitz.

Extremal Functions The fact that a function satisfying (3.1), in both the complete or incomplete case, admits a continuous extension to the closed disk motivates the definition of extremal function for the criterion. This is a function satisfying (3.1) which is not injective on $\partial\mathbf{D}$. These are the analogues for the general criterion of the logarithm function L. An analysis of the geometry of extremal functions depends on being able to join by a geodesic in $\mathbf{D}$ the points on $\partial\mathbf{D}$ where injectivity is lost. We call this an *extremal geodesic*. Again, we need to assume that such joining of boundary points is always possible, and we adopt the following definition:

Definition The metric g on $\mathbf{D}$ has the *Boundary Points Joined* property (BPJ) if any two points on $\partial\mathbf{D}$ can be joined by a geodesic which lies in $\mathbf{D}$ except for its endpoints.

We assume again that the metric g is complete. Suppose f is extremal, with $f(\zeta_1) = f(\zeta_2)$, $|\zeta_1| = |\zeta_2| = 1$. Via a Möbius transformation we may assume that the common value of f is the point at infinity. But then the associated function u_f must be constant along the geodesic γ joining ζ_1, ζ_2, for otherwise, by the finiteness of, say, (3.5), one of $f(\zeta_1)$, $f(\zeta_2)$ would be finite. Some technical considerations imply that the constant value of u_f along the extremal geodesic is its absolute minimum. Therefore, the gradient of u_f vanishes identically along γ, which provides enough information to characterize the extremal function along the curve. In particular, one shows that equality must hold in (3.3) along γ. Thus, for example, if strict inequality holds in (3.3), then the image is a Jordan domain.

We want to emphasize this last point in connection with our use of the term 'extremal'. An extremal function satisfies (3.3) (or (3.1) in the incomplete case) with equality along an extremal geodesic. However, equality can hold in the criterion along some geodesic without the function being an extremal. For example, for Nehari's criterion (2.1) the interval $(-1, 1)$ is an extremal geodesic for the (extremal) function

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

But we also have $|Sf(z)| = 2/(1-|z|^2)^2$ along $(-1, 1)$ for the function

$$f(z) = \frac{1}{\sqrt{2}}\,\frac{(1+z)^{\sqrt{2}} - (1-z)^{\sqrt{2}}}{(1+z)^{\sqrt{2}} + (1-z)^{\sqrt{2}}}, \quad Sf(z) = \frac{-2}{(1-z^2)^2},$$

and $f(\mathbf{D})$ is a Jordan domain, in fact a quasidisk. Hence, in our sense, f is not an extremal function for Nehari's criterion.

Finally, a geometrically striking fact is that the image of an extremal geodesic under an extremal function is a *euclidean* circle. It lies in the image domain except for one point, the point on the boundary of the image which is the image of two points on $\partial\mathbf{D}$ where the function is not one-to-one.

Reflections, Homeomorphic, and Quasiconformal Extensions We now construct an extension modeled on the Ahlfors-Weill formula (2.5), (2.6). We can do this in general only when the metric g is complete, which we again assume. Let f satisfy (3.3) and let $\Omega = f(D)$. We define a function ρ on Ω by the equation

$$f^*(\rho^2|dw|^2) = e^{2\sigma}|dz|^2. \tag{3.6}$$

In other words,

$$\rho(f(z))|f'(z)| = e^{\sigma},$$

or

$$\rho(f(z)) = u_f^2(z). \tag{3.7}$$

By (3.6), f is an isometry between the two given metrics, and by (3.7), $\rho^{1/2}$ is convex in the metric $\rho^2|dw|^2$.

Now set

$$\Lambda_\Omega(w) = w + \frac{1}{\partial_w \log \rho}, \tag{3.8}$$

and define an extension, as before, via

$$E_f(z) = f(z) \quad \text{for } |z| \le 1, \Lambda_\Omega(f(\zeta)), \quad \zeta = 1/\bar{z}, \quad \text{for } |z| > 1. \tag{3.9}$$

Λ_Ω is again conformally natural, which, together with the convexity of $\rho^{1/2}$, allows us to proceed with the analysis of the mapping properties of the relfection Λ_Ω as in the case of the Nehari class. The working hypothesis is that for all Möbius shifts of f, the induced convex function $\rho^{1/2}$ has at most one critical point. This implies that Λ_Ω is one-to-one, and that it assumes values in the complement of the closure of Ω. An estimate analogous to (2.9), together with completeness of the metric, implies that Λ_Ω is the identity on $\partial\Omega$. One thus obtains:

Theorem 7 *The mapping E_f in (3.9) is a homeomorphic extension of f if and only if for every Möbius shift the convex function $\rho^{1/2}$ has at most one critical point.*

If for some Möbius shift the convex function possesses two critical points, then the entire geodesic segment joining them must consist of critical points. As in the analysis of extremal functions, this implies equality in (3.3) along some geodesic segment in $\mathbf{D}$. In particular, we can conclude:

Corollary 1 *If strict inequality holds in (3.3) then the extension E_f is a homeomorphism.*

It is natural to ask whether the failure of the extension to be a homeomorphism occurs exactly when the image domain is not Jordan, *i.e.*, when f is an extremal. Such was the case for the Nehari class. The answer is yes, when the metric g is real analytic. But if the metric is just C^∞ there are simple examples showing that Ω can even be equal to $\mathbf{D}$ but with the associated function u_f having a geodesic segment of critical points. In the absence of real analyticity, the vanishing of the gradient along a geodesic *segment* does not force its vanishing along the entire geodesic, *i.e.*, out to the boundary, and this is the necessary condition for the failure of univalence at the endpoints of the geodesic.

Finally we consider the quasiconformality of the extension. As before, it is not difficult to compute its Beltrami coefficient outside $\mathbf{D}$. In absolute value it is

$$\frac{2}{|K(g)|}||\mathcal{S}_g f||_g.$$

This means that the extension E_f is quasiconformal exactly when (3.3) is replaced by the stronger inequality $||\mathcal{S}_g f||_g \le (t/2)|K(g)|$ for some $t < 1$, just as for the Ahlfors-Weill theorem.

There are several interesting questions still to address here, especially in connection with extremal functions. For example, we know something, but not much, about situations where there is a unique extremal function, or a unique extremal geodesic for an extremal function. This is an interesting question. In addition, for a given metric and a given geodesic γ, there is, as mentioned above, a defining equation that characterizes when an f is extremal along γ. This equation provides enough information so that, in principle, one can solve for f along the curve. We would like to understand better when the map f defined on γ extends to a univalent map of $\mathbf{D}$ which satisfies (3.3). By construction it will produce equality in (3.3) along γ.

Techniques for Incomplete Metrics Recall that the full statement (3.1) of the general injectivity criterion allows for incomplete metrics, with the diameter term appearing as part of the bound for the Schwarzian tensor. Several of the results in the previous paragraphs do work for incomplete metrics, for example continuous extension to $\overline{\mathbf{D}}$ and the analysis of properties of extremal functions. What does not go through without modification is the work on homeomorphic and quasiconformal extensions.

This is a genuine issue. For example, in Nehari's original paper with (2.1) he also proved the very simple univalence criterion

$$|Sf(z)| \leq \frac{\pi^2}{2}. \tag{3.10}$$

which, in turn, is actually the easiest case of the general criterion (3.1); take g to be the euclidean metric on $\mathbf{D}$ – curvature 0, diameter 2. Furthermore, it follows from another result of Gehring and Pommerenke, in the same paper cited earlier, that the image of $\mathbf{D}$ by a function satisfying (3.10) will, in fact, be *quasidisk* as long as it is a Jordan domain. Such an f does then have a quasiconformal extension, but it is not the Ahlfors-Weill extension. However, a modified version of Ahlfors-Weill works. The full details are in [2], but we sketch the main points here.

To begin with, the function

$$F(z) = \tan\left(\frac{\pi}{2}z\right)$$

is, up to rotations of $\mathbf{D}$ and Möbius transformations of the image, the only extremal function for (3.10). It also determines a complete, radial, metric on $\mathbf{D}$ by

$$F'(|z|)|dz| = \cos^{-2}\left(\frac{\pi}{2}|z|\right)|dz|. \tag{3.11}$$

The usefulness of considering this metric comes from following lemma, which can be considered as a sharpening of the Sturm comparison theorem.

Lemma 1 (Relative Convexity) *Let u, v, and q be defined in $[0,1)$ and suppose that*

$$u'' + qu \geq 0, \tag{3.12}$$

and that

$$v'' + qv = 0. \tag{3.13}$$

Then the function

$$w = \left(\frac{u}{v}\right) \circ F^{-1}$$

is convex, where F is defined by $F' = 1/v^2$.

Recall from the introduction that a function f with $Sf = 2p$ can be written as the integral

$$f(z) = \int_{z_0}^{z} u^{-2}(\zeta)\, d\zeta,$$

where $u'' + pu = 0$. In the case $|2p| \leq \pi^2/2 = SF = 2q$, the lemma implies that

$$\sqrt{\frac{F'(|z|)}{|f'(z)|}}$$

is convex along rays from the origin, but when the rays are parametrized by arclength of the metric (3.11). In other words, the associated function u_f of f relative to (3.11) is radially convex. A little more effort gives the full convexity on $\mathbf{D}$, and that a function satisfying (3.10) must also satisfy the general criterion (3.3) with the complete metric (3.11). The condition takes the form

$$\left|\zeta^2 Sf(z) + \frac{\pi^2}{4}\tan^2\left(\frac{\pi}{2}|z|\right) + \frac{\pi}{2}\frac{1}{|z|}\tan\left(\frac{\pi}{2}|z|\right) - \frac{\pi^2}{4}\right| \le \frac{\pi^2}{4}\tan^2\left(\frac{\pi}{2}|z|\right) + \frac{\pi}{2}\frac{1}{|z|}\tan\left(\frac{\pi}{2}|z|\right) + \frac{\pi^2}{4}.$$

The maximum principle implies that for nonextremals it must be that

$$|Sf(z)| < \frac{\pi^2}{2},$$

which in turn implies a strict bound in the big inequality, above. Hence by Corollary 1, the gradient extension defined by (3.8) is a homeomorphism. Unfortunately, the reflection, and hence the corresponding extension E_f, will in general fail to be quasiconformal in a neighborhood of $\partial\mathbf{D}$; the magnitude of the complex dilatation can tend to 1.

Here is where one can modify the construction. To overcome the failure of quasiconformality at the boundary, we perturb the metric (3.11), to one of the form

$$F'(|z|)^{\alpha}|dz|,$$

where $\alpha < 1$ must be chosen sufficiently close to 1, and depending on the given nonextremal function f. In terms of f the reflection takes the form

$$\Lambda(z) = f(z) + \frac{2|z|f'(z)}{\frac{\alpha\pi}{2}\tan\left(\frac{\pi}{2}|z|\right) - |z|\frac{f''}{f'}(z)}.$$

The parameter α can be chosen independent of f if the nonextremal function satisfies the stronger inequality $|Sf(z)| \le t\pi^2/2$ for some $t < 1$. Then Λ will give a quasiconformal reflection in the image for any α with $\max\{1/2, t\} < \alpha < 1$.

The full arguments supporting these facts, which apply to more general criteria than (3.10), are rather technical. We refer to [2].

Notes The conditions (ULP) and (BPJ) must be hypotheses in many of our results, but they are not restrictive conditions on a metric. Several sufficient conditions are given in [5].

It would take us too far afield to discuss the important work of C. Epstein. Though in a different direction, it has been very influential on our own work. We refer to his papers [8] and [9].

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